6. Verify Green's theorem in the plane for $\oint(3x^2-8y^2)dx+(4y-6xy)dy$ where C is the curve bounded by $y=\sqrt{x}$ and $y=x^2$. 7

Sol:- By Green's Theorem, we have

$$\int_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy \quad -(1)$$

Here $M=3x^2-8y^2$ and $N=4y-6xy$

$\frac{\partial M}{\partial y} = -16y$ $\qquad$ $\frac{\partial N}{\partial x} = -6y$

$$= \iint_S \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy = \iint_S -6y-(-16y)\,dx\,dy$$

$$= \iint_S (-6y+16y)\,dx\,dy$$

$$= \iint_S 10y\,dx\,dy$$

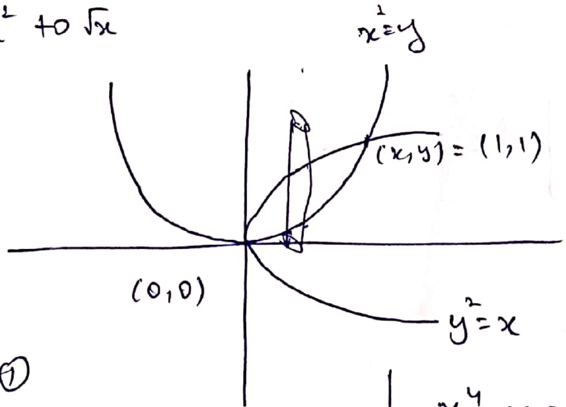
limits are

x: 0 to 1.

y: x^2 to $\sqrt{x}$

x = fix.

$(0,0)$ to $(1,1)$



$x^2=y \quad -(1)$

$y^2=x \quad -(2)$

Subs eqn (1) 'y' value in eqn (2)

$y^2=x$

$(x^2)^2=x \quad \to$

$x^4-x=0$

$x(x^3-1)=0$

$\boxed{x=0}$, $x^3-1=0$

$x^3=1$

$\boxed{x=1}$

x : 0 to 1 $\to$ 'x' limits

for 'y' limits,

$y = x^2$

$y = \sqrt{x}$

in (0,1) interval, substituting

0.5 in place of 'x'

$y = (0.5)^2 = 0.25$ L.L. → 'y' limits

$y = \sqrt{0.5} = 0.7071067.$ U.L

$$\Rightarrow \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 10y \; dx \; dy$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 10y \; dy \; dx$$

$$= \int_{x=0}^{1} \left[\frac{10y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_{x=0} \left[\frac{\cancel{10}^{5}(\sqrt{x})^2}{\cancel{2}} - \frac{\cancel{10}^{5}(x^2)^2}{\cancel{2}}\right] dx$$

$$= \int_{x=0}^{1} 5x - 5x^4 \; dx$$

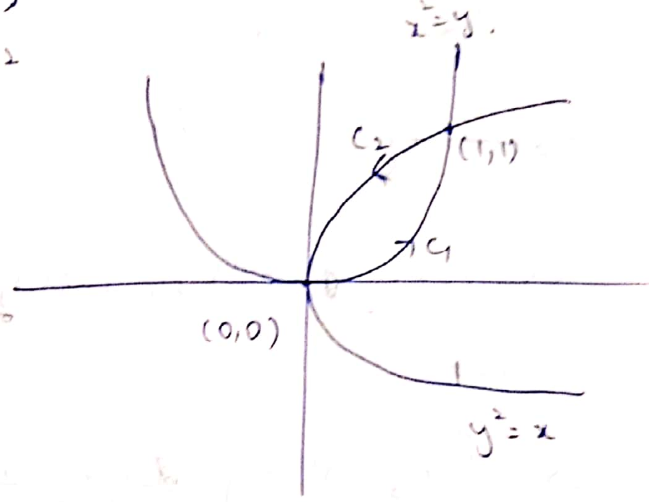
$$= \left[\frac{5x^2}{2}\right]_0^1 - \left[\frac{5x^5}{5}\right]_0^1$$

$$= \frac{5}{2} - 1 = \frac{3}{2} = \text{R.H.S}$$

Now, LHS

$$\oint_C M\,dx + N\,dy = \int_{C_1}(\,) + \int_{C_2}(\,)$$

Along C_1 : (0,0) to (1,1)



$x = 0$ to 1 $\qquad x^2 = y$

$\qquad\qquad 2x\,dx = dy$

$$\Rightarrow \int_{x=0}^{1} (3x^2 - 8\underline{y}^2)\,dx + (4\underline{y} - 6x\underline{y})\,dy$$

$$\Rightarrow \int_{x=0}^{1} (3x^2 - 8(x^2)^2)\,dx + (4(x^2) - 6x(x^2))\cdot(2x\,dx)$$

$$\Rightarrow \int_{x=0}^{1} (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)(2x\,dx)$$

$$\Rightarrow \int_{x=0}^{1} (3x^2 - 8x^4)\,dx + (8x^3 - 12x^4)\,dx$$

$$\Rightarrow \left[\frac{3x^3}{3}\right]_0^1 - \left[\frac{8x^5}{5}\right]_0^1 + \left[\frac{8x^4}{4}\right]_0^1 - \left[\frac{12x^5}{5}\right]_0^1$$

$$= \frac{3}{3} - \frac{8}{5} + \frac{8}{4} - \frac{12}{5}$$

$$= 1 - \frac{8}{5} + 2 - \frac{12}{5}$$

$$= 3 - \frac{\cancel{20}\,^{4}}{\cancel{5}}$$

$$= -1.$$

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Along C_2: $(1,1)$ to $(0,0)$

$\cancel{x^2 = y}$ $\quad y^2 = x \quad\quad \Big| \quad x = 1$ to 0

$\cancel{2x\,dx = dy}$ $\quad 2y\,dy = dx$

$$\Rightarrow \int_{x=1}^{0} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$\Rightarrow \int_{x=1}^{0} (3(y^2)^2 - 8y^2)(2y\,dy) + (4y - 6y(y^2))\,dy$$

$$\Rightarrow \int_{x=1}^{0} (3y^4 - 8y^2)(2y\,dy) + (4y - 6y^3)\,dy$$

$$= \int_{x=0}^{1} (6y^5 - 16y^3)\,dy + (4y - 6y^3)\,dy$$

$$= \left[\frac{6y^6}{6}\right]_1^0 - \left[\frac{16y^4}{4}\right]_1^0 + \left[\frac{4y^2}{2}\right]_1^0 - \left[\frac{\cancel{6}^{3}y^4}{\cancel{4}_{2}}\right]_1^0$$

$$= -\left[\frac{\cancel{6}^{1}}{\cancel{6}} - \frac{\cancel{16}^{4}}{\cancel{4}} + 2 - \frac{3}{2}\right] \Rightarrow -\left[1 - 4 + 2 - \frac{3}{2}\right]$$

$$\Rightarrow -\left[-1 - \frac{3}{2}\right]$$

$$\Rightarrow -\left(-\frac{5}{2}\right)$$

$$\Rightarrow 5/2$$

$$\oint M\,dx + N\,dy = -1 + \frac{5}{2}$$

$$= \frac{3}{2}$$

$\therefore$ L.H.S = R.H.S